On smoothing and on reducing variance of statistical functionals.

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Let $T(F)$ be a function of the cumulative distribution function (cdf), often called a statistical functional. Empirical statistical functionals $T\left(\hat{F}_n\right)$, where $\hat{F}_n$ is the empirical cdf based on a sample with a cdf F, are frequently used to estimate $T(F)$. Alternatively, many authors (cf. Falk & Reiss (1989), Cheng & Parzen (1997), Fernholz (1993), Fernholz (1997)) consider $T\left(\tilde{F}_{k_n,n}\right)$ as estimators of $T(F)$, where $\tilde{F}_{k_n,n}$ is a kernel smoothed estimator of the cdf F, with the aim to reduce the asymptotic variance of the estimator. The smoothing introduces some bias, therefore the window width h_n in the kernel k used in $\tilde{F}_{k_n,n}$ is being made dependent on the sample size n and converges to zero. As noted in Fernholz (1997), by decreasing the window width k_n one also reduces possible gains in the asymptotic variance of the smoothed estimators. In this paper we consider another strategy to retain the reduction of the asymptotic variance and at the same time to reduce the asymptotic bias.

We consider several estimators $T\left(\tilde{F}_{k_{h_i},n}\right)$ with window width $h_1, \ldots, h_k$ not varying with the sample size. Next, based on the asymptotic representation of $T(F * k_h)$ for small $h > 0$, we consider a polynomial approximation of $T(F * k_h)$.

We suggest the estimator of the intercept, based on $T\left(\tilde{F}_{k_{h_i},n}\right)$, $i = 1 \ldots, k$ to be used as an estimator of $T(F)$. We show that, under proper regularity assumptions, it is asymptotically unbiased and has asymptotic variance lower than that of the empirical functional $T\left(\hat{F}_n\right)$.

References

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